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The Schouten–Nijenhuis bracket, cohomology and generalized Poisson structures

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Abstract. Newly introduced generalized Poisson structures based on suitable skew-symmetric contravariant tensors of even order are discussed in terms of the Schouten–Nijenhuis bracket. The associated 'Jacobi identities' are expressed as conditions on these tensors, the cohomological contents of which is given. In particular, we determine the linear generalized Poisson structures which can be constructed on the dual spaces of simple Lie algebras.

1. Introduction

In 1973 Nambu [1] proposed a generalization of the standard classical Hamiltonian mechanics based on a three-dimensional 'phase space' spanned by a canonical triplet of dynamical variables and on two 'Hamiltonians'. His approach was later discussed by Bayen and Flato [2] and, e.g., in [3–5]. Recently, a higher-order extension of Nambu's approach, involving $(n-1)$ Hamiltonians, was proposed by Takhtajan [6] (see [7] for applications). This approach, which includes Nambu's mechanics as a particular case, has the property that the time derivative is a derivation of the nth-order Poisson bracket (PB) because the expression of this fact, which involves $(n+1)$ terms, is the same as the 'fundamental identity' [6] which generalizes the Jacobi identity of the ordinary $n=2$ case. Closely related to Hamiltonian dynamics is the study of Poisson structures (PS) (see [8–10]) on a manifold M.

Recently, a different generalization of PS has been put forward [11]. In contrast to those of Nambu and Takhtajan, the dynamics is associated with generalized Poisson brackets (GPB) necessarily involving an even number of functions. The aim of this paper is to discuss these new generalized Poisson structures (GPS) further and, in particular, to exhibit the cohomological contents of the examples provided (for the linear GPS) on the dual spaces of simple Lie algebras. The key idea of the new GPS is the replacement of the skew-symmetric bivector Λ defining the standard Poisson structure by appropriate skew-symmetric contravariant tensor fields of even order $\Lambda^{(2p)}$. For a standard $(p=1)$ PS, the property which guarantees the Jacobi identity for the PB of two functions on a Poisson manifold may be expressed [8, 12] as $[\Lambda, \Lambda]=0$, where $\Lambda \equiv \Lambda^{(2)}$ is the bivector field which may be used to define the Poisson structure and $[\ ,\ ]$ is the Schouten–Nijenhuis

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bracket (SNB) [13, 14]. Thus, a natural generalization of the standard PS may be found [11] using $\Lambda^{(2p)}$, and replacing the Jacobi identity by the condition which follows from $[\Lambda^{(2p)}, \Lambda^{(2p)}] = 0$. The vanishing of the SNB of $\Lambda^{(2p)}$ with itself generalizes the Jacobi identity in a geometrical way that is different from [1, 6]: our GPB involve an *even* number of functions, whereas this number is arbitrary (three in [1]) in earlier extensions.

The geometrical content of the theory becomes especially apparent when the linear GPS on the duals $\mathcal{G}^*$ of simple Lie algebras $\mathcal{G}$ are considered, since these automatically provide us with solutions of the generalized Jacobi identities (GJI) that our Poisson structures must satisfy. In fact, since the Jacobi identity or its generalizations constitute the only essentially non-trivial ingredient of any PS (the skew-symmetry and the Leibniz rule are easy to satisfy), it is important to have explicit examples which satisfy them. In our linear GPS, the solution to the GJI has a cohomological character: the different tensors $\Lambda^{(2p)}$ that can be introduced are related to Lie algebra cocycles.

2. Standard Poisson structures

Let us recall for completeness some facts concerning standard PS. Let M be a manifold and $\mathcal{F}(M)$ be the associative algebra of smooth functions on M.

Definition 2.1 (PB). A *Poisson bracket* $\{\cdot,\cdot\}$ on $\mathcal{F}(M)$ is a bilinear mapping assigning to every pair of functions $f_1, f_2 \in \mathcal{F}(M)$ a new function $\{f_1, f_2\} \in \mathcal{F}(M)$, with the following conditions:

(a) skew-symmetry

$$\{f_1, f_2\} = -\{f_2, f_1\} , \tag{2.1}$$

(b) Leibniz rule (derivation property)

$$\{f, gh\} = g\{f, h\} + \{f, g\}h , \tag{2.2}$$

(c) Jacobi identity (JI)

$$\tfrac{1}{2}\,\mathrm{Alt}\,\{f_1, \{f_2, f_3\}\} \equiv \{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0. \tag{2.3}$$

M is then called a *Poisson manifold*. Because of (2.1), (2.3) the space $\mathcal{F}(M)$ endowed with the PB $\{\cdot,\cdot\}$ becomes an (infinite-dimensional) Lie algebra.

Let x^j be local coordinates on $U \subset M$ and consider a PB of the form

$$\{f(x), g(x)\} = \omega^{jk}(x)\partial_j f \partial_k g , \qquad \partial_j = \frac{\partial}{\partial x^j} , \qquad j, k = 1, \ldots, n = \dim M . \tag{2.4}$$

Then $\omega^{ij}(x)$ defines a PB if $\omega^{ij}(x) = -\omega^{ji}(x)$ (equation (2.1)) and (equation (2.3))

$$\omega^{jk}\partial_k\omega^{lm} + \omega^{lk}\partial_k\omega^{mj} + \omega^{mk}\partial_k\omega^{jl} = 0 . \tag{2.5}$$

The requirements (2.1) and (2.2) indicate that the PB may be given in terms of a skew-symmetric bivector field (*Poisson bivector*) $\Lambda \in \wedge^2(M)$ which is uniquely defined. Locally,

$$\Lambda = \tfrac{1}{2}\omega^{jk}\partial_j \wedge \partial_k . \tag{2.6}$$

Condition (2.5) is taken into account by requiring $[\Lambda, \Lambda] = 0$ [8, 12] (section 3). Then Λ defines a *Poisson structure* on M and the PB is defined by

$$\{f, g\} = \Lambda(df, dg) , \qquad f, g \in \mathcal{F}(M) . \tag{2.7}$$

Definition 2.2. Let $H(x) \in \mathcal{F}(M)$. Then the vector field $X_H = i_{dH}\Lambda$ (where $i_\alpha\Lambda(\beta) := \Lambda(\alpha, \beta)$, α, β one-forms), is called a *Hamiltonian vector field* of H.

From the JI, equation (2.3), it easily follows that

$$[X_f, X_H] = X_{\{f,H\}} . \tag{2.8}$$

Thus, the Hamiltonian vector fields form a Lie subalgebra of the Lie algebra $\mathcal{X}(M)$ of all smooth vector fields on M. Locally

$$X_H(x) = \omega^{jk}(x)(\partial_j H(x))\partial_k \; ; \qquad X_H.f = \{H, f\} . \tag{2.9}$$

We recall that the tensor $\omega^{jk}(x)$ appearing in (2.4), (2.6) does not need to be non-degenerate; in particular, the dimension of a Poisson manifold M may be odd. Only when Λ has constant rank $2q$ (i.e. it is *regular*) and the codimension ($\dim M - 2q$) of the manifold is zero does Λ define a *symplectic structure*.

3. Standard linear Poisson structures

A particular class of Poisson structures is that defined on the duals $\mathcal{G}^*$ of the Lie algebras $\mathcal{G}$. The case of the linear Poisson structures was considered by Lie himself [15, 16], and has been further investigated recently [17–19]. Let $\mathcal{G}$ be a real finite-dimensional Lie algebra $\mathcal{G}$ with Lie bracket $[\, .\, , \, .\,]$. The natural identification $\mathcal{G} \cong (\mathcal{G}^*)^*$, allows us to think of $\mathcal{G}$ as a subspace of linear functions of the ring of smooth functions $\mathcal{F}(\mathcal{G}^*)$. Choosing a basis $\{\, e_i \,\}_{i=1}^r$ of $\mathcal{G}$, $[e_i, e_j] = C_{ij}^k e_k$, and identifying its elements with linear coordinate functions x_i on the dual space $\mathcal{G}^*$ by means of $x_i(x) = \langle x, e_i \rangle$ for all $x \in \mathcal{G}^*$, the fundamental PB on $\mathcal{G}^*$ may be defined in a natural way by taking

$$\{x_i, x_j\}_{\mathcal{G}} = C_{ij}^k x_k = \omega_{ij}(x) \; , \qquad i, j, k = 1, \ldots, r = \dim \mathcal{G} \; , \tag{3.1}$$

since the Jacobi identity for C_{ij}^k implies that (2.5) is satisfied. Intrinsically, the PB $\{.,.\}_{\mathcal{G}}$ on $\mathcal{F}(\mathcal{G}^*)$ is defined by

$$\{f, g\}_{\mathcal{G}}(x) = \langle x, [df(x), dg(x)]\rangle \; , \qquad f, g \in \mathcal{F}(\mathcal{G}^*), \;\; x \in \mathcal{G}^* \; , \tag{3.2}$$

where the one-forms in the bracket are regarded as linear mappings from $T_x(\mathcal{G}^*) \sim \mathcal{G}^*$ to $\mathbb{R}$ and hence as elements of $\mathcal{G}$. Locally,

$$[df(x), dg(x)] = e_k C_{ij}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \; , \qquad \{f, g\}_{\mathcal{G}}(x) = x_k C_{ij}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \, . \tag{3.3}$$

The above PB $\{.,.\}_{\mathcal{G}}$ (see [20]) is commonly called a *Lie–Poisson bracket* and defines a *Lie–Poisson* structure on $\mathcal{G}^*$. It is associated with the bivector field $\Lambda_{\mathcal{G}}$ on $\mathcal{G}^*$ locally written as

$$\Lambda_{\mathcal{G}} = \frac{1}{2} C_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \equiv \frac{1}{2}\omega_{ij}\partial^i \wedge \partial^j \tag{3.4}$$

(cf equation (2.6)), so that (cf equation (2.7)) $\Lambda_{\mathcal{G}}(df, dg) = \{f, g\}_{\mathcal{G}}$. Then $[\Lambda_{\mathcal{G}}, \Lambda_{\mathcal{G}}] = 0$ (cf equation (2.5)) leads to the Jacobi identity for $\mathcal{G}$, which may be written as

$$\tfrac{1}{2}\,\mathrm{Alt}\,(C_{i_1 i_2}^{\rho} C_{\rho i_3}^{\sigma}) \equiv \tfrac{1}{2}\epsilon_{i_1 i_2 i_3}^{j_1 j_2 j_3} C_{j_1 j_2}^{\rho} C_{\rho j_3}^{\sigma} = 0 \, . \tag{3.5}$$

Note also that the Poisson bracket of two polynomial functions on $\mathcal{G}^*$ is again a polynomial function, so that the space $\mathcal{P}(\mathcal{G}^*)$ of all polynomials on $\mathcal{G}^*$ is a Lie subalgebra.

Let β be a closed one form on $\mathcal{G}^*$. The associated vector field

$$X_\beta = i_\beta \Lambda_{\mathcal{G}} \; , \tag{3.6}$$

is an infinitesimal automorphism of $\Lambda_{\mathcal{G}}$, i.e.

$$L_{X_\beta}\Lambda_{\mathcal{G}} = 0 \,, \tag{3.7}$$

and $[X_f, X_g] = X_{\{f,g\}}$ (equation (2.8)); this is proved easily using that $L_{X_f} g = \{f, g\}$ and $L_{X_f}\Lambda_{\mathcal{G}} = 0$. It follows from (3.4) that the Hamiltonian vector fields $X_i = i_{dx_i}\Lambda_{\mathcal{G}}$ corresponding to the linear coordinate functions x_i, have the expression (cf equation (2.9))

$$X_i = C_{ij}^k x_k \frac{\partial}{\partial x_j} \,, \qquad i = 1, \ldots, r = \dim \mathcal{G} \tag{3.8}$$

so that the Poisson bivector can be written as

$$\Lambda_{\mathcal{G}} = -\frac{1}{2} X_i \wedge \frac{\partial}{\partial x_i} \,. \tag{3.9}$$

Note that this way of writing $\Lambda_{\mathcal{G}}$ is not unique. Using the adjoint representation of $\mathcal{G}$, $(C_i)^k_{\cdot j} = C^k_{ij}$ the Poisson bivector $\Lambda_{\mathcal{G}}$ may be rewritten as

$$\Lambda_{\mathcal{G}} = -\frac{1}{2} X_{C_i} \wedge \frac{\partial}{\partial x_i} \qquad \left(X_{C_i} = x_k (C_i)^k_{\cdot j} \frac{\partial}{\partial x_j} \right) . \tag{3.10}$$

The vector fields X_{C_i} provide a realization of ad $\mathcal{G}$ in terms of vector fields on $\mathcal{G}^*$.

4. The Schouten–Nijenhuis bracket

Let $\wedge(M) = \bigoplus_{j=0}^{n} \wedge^j(M)$ ($\wedge^0 = \mathcal{F}(M)$, $n = \dim M$), be the contravariant exterior algebra of skew-symmetric contravariant (i.e. tangent) tensor fields (*multivectors* or *j-vectors*) over M. The Lie bracket of vector fields on M may be uniquely extended to an $\mathbb{R}$-bilinear bracket on $\wedge(M)$, the SNB, in such way that $\wedge(M)$ becomes a graded superalgebra (see the remark below). The SNB [13, 14] is a bilinear mapping $\wedge^p(M) \times \wedge^q(M) \to \wedge^{p+q-1}(M)$. We start by defining the SNB for multivectors given by products of vector fields.

Definition 4.1. Let $X_1, \ldots, X_p, Y_1, \ldots, Y_q$ be vector fields over M. Then

$$[X_1 \wedge \cdots \wedge X_p \,,\; Y_1 \wedge \cdots \wedge Y_q]$$
$$= \sum (-1)^{t+s} X_1 \wedge \cdots \widehat{X}_s \cdots \wedge X_p \wedge [X_s, Y_t] \wedge Y_1 \wedge \cdots \widehat{Y}_t \cdots \wedge Y_q \,, \tag{4.1}$$

where [,] is the SNB and $\widehat{X}$ stands for the omission of X. It is easy to check that (4.1) is equivalent to original definition [13, 14].

Theorem 4.1. Let $M = G$ be the group manifold of a Lie group, and let the above vector fields X, Y be left-invariant (LI) (right-invariant (RI)) vector fields on G. Then the SNB of LI (RI) skew multivector fields is also LI (RI).

Proof. It suffices to recall that if X is LI, $L_Z X = [Z, X] = 0$ where Z is the generator of the left translations. □

Definition 4.2. Let $A \in \wedge^p(M)$ and $B \in \wedge^q(M)$, $p, q \leqslant n$, be the p- and q-vectors (multivectors of order p and q, respectively) given in a local chart by

$$A(x) = \frac{1}{p!} A^{i_1 \ldots i_p}(x) \partial_{i_1} \wedge \cdots \wedge \partial_{i_p} \,, \qquad B(x) = \frac{1}{q!} B^{j_1 \ldots j_q}(x) \partial_{j_1} \wedge \cdots \wedge \partial_{j_q} \,. \tag{4.2}$$

The SNB of A and B is the skew-symmetric contravariant tensor field $[A, B] \in \wedge^{p+q-1}(M)$

$$[A, B] = \frac{1}{(p+q-1)!}[A, B]^{k_1 \ldots k_{p+q-1}} \partial_{k_1} \wedge \cdots \wedge \partial_{k_{p+q-1}} \ ,$$

$$[A, B]^{k_1 \ldots k_{p+q-1}} = \frac{1}{(p-1)!q!} \epsilon^{k_1 \ldots k_{p+q-1}}_{i_1 \ldots i_{p-1} j_1 \ldots j_q} A^{\nu i_1 \ldots i_{p-1}} \partial_\nu B^{j_1 \ldots j_q} + \frac{(-1)^p}{p!(q-1)!} \epsilon^{k_1 \ldots k_{p+q-1}}_{i_1 \ldots i_p j_1 \ldots j_{q-1}} B^{\nu j_1 \ldots j_{q-1}} \partial_\nu A^{i_1 \ldots i_p} \ , \tag{4.3}$$

where ϵ is the antisymmetric Kronecker symbol

$$\epsilon^{i_1 \ldots i_p}_{j_1 \ldots j_p} = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_p} \\ \vdots & & \vdots \\ \delta^{i_p}_{j_1} & \cdots & \delta^{i_p}_{j_p} \end{pmatrix} . \tag{4.4}$$

The SNB is graded-commutative:

$$[A, B] = (-1)^{pq} [B, A] . \tag{4.5}$$

As a result, the SNB is identically zero if $A = B$ are of odd order (or even *degree*; $\mathrm{degree}(A) \equiv \mathrm{order}(A) - 1$). It satisfies the graded Jacobi identity

$$(-1)^{pr} [[A, B], C] + (-1)^{qp} [[B, C], A] + (-1)^{rq} [[C, A], B] = 0 \ , \tag{4.6}$$

where (p, q, r) denote the order of (A, B, C), respectively (thus, if Λ is of even order and $[\Lambda, \Lambda] = 0$ it follows from (4.6) that $[\Lambda, [\Lambda, C]] = 0$).

Let $A \wedge B \in \wedge^{p+q}(M)$,

$$(A \wedge B) = \frac{1}{(p+q)!}(A \wedge B)^{i_1 \ldots i_{p+q}} \partial_{i_1} \wedge \cdots \wedge \partial_{i_{p+q}} \ ,$$
$$(A \wedge B)^{i_1 \ldots i_{p+q}} = \frac{1}{p!q!} \epsilon^{i_1 \ldots i_{p+q}}_{j_1 \ldots j_{p+q}} A^{j_1 \ldots j_p} B^{j_{p+1} \ldots j_{p+q}} \ , \tag{4.7}$$

and let $\alpha \in \wedge_{p+q-1}(M)$ be an arbitrary $(p+q-1)$-form, $\alpha = (1/(p+q-1)!)\alpha_{i_1 \ldots i_{p+q-1}} dx^{i_1} \wedge \cdots \wedge dx^{i_{p+q-1}}$. Then the well known formula for one-forms and vector fields, $d\omega(X, Y) = L_X \omega(Y) - L_Y \omega(X) - i_{[X,Y]}\omega$, generalizes to

$$i_{A \wedge B} d\alpha = (-1)^{pq+q} i_A d(i_B \alpha) + (-1)^p i_B d(i_A \alpha) - i_{[A,B]}\alpha \ , \tag{4.8}$$

where the contraction $i_A \alpha$ is the $(q-1)$-form

$$i_A \alpha(\cdot) = \frac{1}{p!}\alpha(A, \cdot) \ , \qquad i_A \alpha = \frac{1}{(q-1)!} \frac{1}{p!} A^{i_1 \ldots i_p} \alpha_{i_1 \ldots i_p j_1 \ldots j_{q-1}} dx^{j_1} \wedge \cdots \wedge dx^{j_{q-1}} \ , \tag{4.9}$$

so that, on forms, $i_B i_A = i_{A \wedge B}$. When α is *closed*, equation (4.8) provides a definition of the SNB through $i_{[A,B]}\alpha$.

From the definition of the SNB it follows that

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(p-1)q} B \wedge [A, C] \ , \tag{4.10}$$

$$[A \wedge B, C] = (-1)^p A \wedge [B, C] + (-1)^{rq} [A, C] \wedge B \ . \tag{4.11}$$

In particular, for the case of the SNB between the wedge products of two *vector* fields

$$[A \wedge B, X \wedge Y] = -A \wedge [B, X] \wedge Y + B \wedge [A, X] \wedge Y - B \wedge [A, Y] \wedge X + A \wedge [B, Y] \wedge X \ , \tag{4.12}$$

so that

$$[A \wedge B, A \wedge B] = -2A \wedge B \wedge [A, B]. \tag{4.13}$$

For instance, if Λ is given by (3.9), we may apply (4.12) to find that the condition $[\Lambda, \Lambda] = 0$ leads to the Jacobi identity.

Remark. It is worth mentioning that the SNB is the unique (up to a constant) extension of the usual Lie bracket of vector fields (see also theorem 4.1) which makes a Z_2-graded Lie algebra of the (graded-)commutative algebra of skew-symmetric contravariant tensors: degree$([A, B])$ = degree(A) + degree(B). In it, the adjoint action is a graded derivation with respect to the wedge product [21] (see equation (4.10)). To make this graded structure explicit, it is convenient to define a new SNB, $[\ ,\ ]'$, which differs from the original one $[\ ,\ ]$ by a factor $(-1)^{p+1}$ on the right-hand side of (4.1), (4.3):

$$[A, B]' := (-1)^{p+1}[A, B]. \tag{4.14}$$

This definition modifies (4.5) to read

$$[A, B]' = -(-1)^{(p-1)(q-1)}[B, A]' \equiv -(-1)^{ab}[B, A]' \tag{4.15}$$

where a = degree$(A) = (p-1)$, etc. Similarly, equation (4.6) is replaced by

$$(-1)^{pr+q+1}[[A, B]', C]' + (-1)^{qp+r+1}[[B, C]', A]' + (-1)^{rq+p+1}[[C, A]', B'] = 0\ , \tag{4.16}$$

which in terms of the degrees (a, b, c) of A, B, C adopts the graded JI form

$$(-1)^{ac}[[A, B]', C]' + (-1)^{ba}[[B, C]', A]' + (-1)^{cb}[[C, A]', B'] = 0. \tag{4.17}$$

The definition (4.14) is used in [17–19, 21] and more adequately stresses the graded structure of the exterior algebra of skew multivector fields; for instance, equations (4.15) and (4.17) have the same form as in supersymmetry (see, e.g., [22]). In this paper, however, we shall use definition 4.2 for the SNB, as in [8, 10, 14] and others.

Definition 4.3. A bivector $\Lambda \in \wedge^2(M)$ is called a *Poisson bivector* and defines a PS on M (and a Poisson bracket on $\mathcal{F}(M) \times \mathcal{F}(M)$) if it commutes with itself under the SNB

$$[\Lambda, \Lambda] = 0 \tag{4.18}$$

(for the case of linear PS this is equivalent to the classical Yang–Baxter equation). Two Poisson bivectors Λ_1, Λ_2 are called *compatible* if the SNB between themselves is zero,

$$[\Lambda_1, \Lambda_2] = 0. \tag{4.19}$$

The compatibility condition is equivalent to requiring that any linear combination $\lambda\Lambda_1 + \mu\Lambda_2$ be a Poisson bivector.

5. Generalized Poisson structures

Since equations (2.1) and (2.2) are automatic for a bivector field, the only stringent condition that a $\Lambda \equiv \Lambda^{(2)}$ defining a PS must satisfy is the Jacobi identity (2.3) or, equivalently, (4.18). It is then natural to consider generalizations of the standard PS in terms of $2p$-ary operations determined by skew-symmetric $2p$-vector fields $\Lambda^{(2p)}$, the case $p = 1$ being the standard one. Since the SNB vanishes identically if Λ' is of odd order (equation (4.5)), only $[\Lambda', \Lambda'] = 0$ for Λ' of even order (odd *degree*) will be non-empty.

Having this in mind, let us first introduce the generalized Poisson bracket (GPB).

Definition 5.1. A *generalized Poisson bracket* $\{\cdot,\cdot,\ldots,\cdot,\cdot\}$ on M is a mapping $\mathcal{F}(M)\times\overset{2p}{\cdots}\times\mathcal{F}(M)\to\mathcal{F}(M)$ assigning a function $\{f_1, f_2, \ldots, f_{2p}\}$ to every set $f_1,\ldots,f_{2p}\in\mathcal{F}(M)$ which is linear in all arguments and satisfies the following conditions:
(a) complete skew-symmetry in f_j;
(b) Leibniz rule: $\forall f_i, g, h \in \mathcal{F}(M)$,

$$\{f_1, f_2, \ldots, f_{2p-1}, gh\} = g\,\{f_1, f_2, \ldots, f_{2p-1}, h\} + \{f_1, f_2, \ldots, f_{2p-1}, g\}h\ ; \tag{5.1}$$

(c) generalized Jacobi identity: $\forall f_i \in \mathcal{F}(M)$,

$$\frac{1}{(2p-1)!}\frac{1}{(2p)!}\,\mathrm{Alt}\,\{f_1, f_2, \ldots, f_{2p-1}, \{f_{2p}, \ldots, f_{4p-1}\}\} = 0\,. \tag{5.2}$$

Conditions (a) and (b) imply that our GPB is given by a skew-symmetric multiderivative, i.e. by a completely skew-symmetric $2p$-vector field $\Lambda^{(2p)} \in \wedge^{2p}(M)$. Condition (5.2) (different from the generalization in [1, 6]) will be called the *generalized Jacobi identity* (GJI); for $p = 2$ it contains 35 terms† (C^{2p-1}_{4p-1} in the general case). It may be rewritten as $[\Lambda^{(2p)}, \Lambda^{(2p)}] = 0$ which, due to (4.5), is not identically zero and gives a non-trivial condition; $\Lambda^{(2p)}$ defines a GPB. We shall see in section 8 that in the linear case our generalized PS are automatically obtained from *constant* skew-symmetric tensors of order $2p+1$. Clearly, the above relations reproduce the ordinary case (2.1)–(2.3) for $p = 1$. The compatibility condition in definition 4.3 may be now extended in the following sense: two GPS $\Lambda_1^{(2p)}$ and $\Lambda_2^{(2q)}$ on M are called *compatible* if they 'commute', i.e. if $[\Lambda_1^{(2p)}, \Lambda_2^{(2q)}] = 0$ (of course, if $p \neq q$ the sum of $\Lambda_1^{(2p)}$ and $\Lambda_2^{(2q)}$ is not defined).

In local coordinates the GPB has the form

$$\{f_1(x), f_2(x), \ldots, f_{2p}(x)\} = \omega_{j_1 j_2 \ldots j_{2p}}(x)\partial^{j_1} f_1\, \partial^{j_2} f_2 \cdots \partial^{j_{2p}} f_{2p} \tag{5.3}$$

where $\omega_{j_1 j_2 \ldots j_{2p}}$ are the coordinates of a completely skew-symmetric tensor which, as a result of (5.2), satisfies

$$\mathrm{Alt}\,(\omega_{j_1 j_2 \ldots j_{2p-1} k}\, \partial^k\, \omega_{j_{2p} \ldots j_{4p-1}}) = 0\,. \tag{5.4}$$

† Explicitly, the $p = 2$ GJI has the form

$$\begin{aligned}
&\{f_1, f_2, f_3, \{f_4, f_5, f_6, f_7\}\} - \{f_4, f_2, f_3, \{f_1, f_5, f_6, f_7\}\} - \{f_1, f_4, f_3, \{f_2, f_5, f_6, f_7\}\}\\
&\quad -\{f_1, f_2, f_4, \{f_3, f_5, f_6, f_7\}\} - \{f_5, f_2, f_3, \{f_4, f_1, f_6, f_7\}\} - \{f_1, f_5, f_3, \{f_4, f_2, f_6, f_7\}\}\\
&\quad -\{f_1, f_2, f_5, \{f_4, f_3, f_6, f_7\}\} - \{f_6, f_2, f_3, \{f_4, f_5, f_1, f_7\}\} - \{f_1, f_6, f_3, \{f_4, f_5, f_2, f_7\}\}\\
&\quad -\{f_1, f_2, f_6, \{f_4, f_5, f_3, f_7\}\} - \{f_7, f_2, f_3, \{f_4, f_5, f_6, f_1\}\} - \{f_1, f_7, f_3, \{f_4, f_5, f_6, f_2\}\}\\
&\quad -\{f_1, f_2, f_7, \{f_4, f_5, f_6, f_3\}\} + \{f_4, f_5, f_3, \{f_1, f_2, f_6, f_7\}\} + \{f_4, f_2, f_5, \{f_1, f_3, f_6, f_7\}\}\\
&\quad +\{f_1, f_4, f_5, \{f_2, f_3, f_6, f_7\}\} + \{f_4, f_6, f_3, \{f_1, f_5, f_2, f_7\}\} + \{f_4, f_2, f_6, \{f_1, f_5, f_3, f_7\}\}\\
&\quad +\{f_1, f_4, f_6, \{f_2, f_5, f_3, f_7\}\} + \{f_4, f_7, f_3, \{f_1, f_5, f_6, f_2\}\} + \{f_4, f_2, f_7, \{f_1, f_5, f_6, f_3\}\}\\
&\quad +\{f_1, f_4, f_7, \{f_2, f_5, f_6, f_3\}\} + \{f_5, f_6, f_3, \{f_4, f_1, f_2, f_7\}\} + \{f_5, f_2, f_6, \{f_4, f_1, f_3, f_7\}\}\\
&\quad +\{f_1, f_5, f_6, \{f_4, f_2, f_3, f_7\}\} + \{f_5, f_7, f_3, \{f_4, f_1, f_6, f_2\}\} + \{f_5, f_2, f_7, \{f_4, f_1, f_6, f_3\}\}\\
&\quad +\{f_1, f_5, f_7, \{f_4, f_2, f_6, f_3\}\} + \{f_6, f_7, f_3, \{f_4, f_5, f_1, f_2\}\} + \{f_6, f_2, f_7, \{f_4, f_5, f_1, f_3\}\}\\
&\quad +\{f_1, f_6, f_7, \{f_4, f_5, f_2, f_3\}\} - \{f_4, f_5, f_6, \{f_1, f_2, f_3, f_7\}\} - \{f_4, f_5, f_7, \{f_1, f_2, f_6, f_3\}\}\\
&\quad -\{f_4, f_6, f_7, \{f_1, f_5, f_2, f_3\}\} - \{f_5, f_6, f_7, \{f_4, f_1, f_2, f_3\}\} = 0\,.
\end{aligned}$$

Definition 5.2. A skew-symmetric $2p$-vector field $\Lambda^{(2p)} \in \wedge^{(2p)}(M)$, locally written as

$$\Lambda^{(2p)} = \frac{1}{(2p)!}\, \omega_{j_1 \ldots j_{2p}}\, \partial^{j_1} \wedge \cdots \wedge \partial^{j_{2p}} \ , \tag{5.5}$$

defines a *generalized Poisson structure iff* $[\Lambda^{(2p)}, \Lambda^{(2p)}] = 0$, which reproduces equation (5.4).

6. Generalized dynamics

Let us now introduce a dynamical system associated with the above generalized Poisson structure. Namely, let us fix a set of $(2p-1)$ 'Hamiltonian' functions $H_1, H_2, \ldots, H_{2p-1}$. The time evolution of x_j, $f \in \mathcal{F}(M)$ is defined by

$$\dot{x}_j = \{H_1, \ldots, H_{2p-1}, x_j\} \ , \qquad \dot{f} = \{H_1, \ldots, H_{2p-1}, f\} \, . \tag{6.1}$$

Definition 6.1. The *Hamiltonian vector field* associated with the $(2p-1)$ Hamiltonians $H_1, \ldots, H_{2p-1}$ is defined by $X_{H_1,\ldots,H_{2p-1}} = i_{dH_1 \wedge \cdots \wedge dH_{2p-1}} \Lambda$. Thus,

$$\begin{aligned}(i_{dH_1 \wedge \cdots \wedge dH_{2p-1}} \Lambda)_j &= \frac{1}{(2p-1)!} \Lambda(dH_1 \wedge \cdots \wedge dH_{2p-1}, dx_j) \\ &= \Lambda(dH_1, \ldots, dH_{2p-1}, dx_j) \ ; \end{aligned} \tag{6.2}$$

$$X_{H_1,\ldots,H_{2p-1}} = \omega_{i_1 \ldots i_{2p-1} j} \partial^{i_1} H_1 \cdots \partial^{i_{2p-1}} H_{2p-1} \partial^j \ .$$

Definition 6.2. The *generalized Hamiltonian system* is defined by the equation

$$\dot{x}_j = X_j = (X_{H_1,\ldots,H_{2p-1}})_j = \omega_{i_1 \ldots i_{2p-1} j} \partial^{i_1} H_1 \cdots \partial^{i_{2p-1}} H_{2p-1} \, . \tag{6.3}$$

Then $\dot{f} = X_{H_1,\ldots,H_{2p-1}} \cdot f \ (= \dot{x}_j \partial f / \partial x_j)$ is given by (6.1).

Definition 6.3. A function $f \in \mathcal{F}(M)$ is a *constant of the motion* if (6.1) is zero.

Due to the skew-symmetry of the GPB, the Hamiltonian functions $H_1, \ldots, H_{2p-1}$ are all constants of the motion but the system may have additional ones $h_{2p}, \ldots, h_k$; $k \geqslant 2p$.

Definition 6.4. A set of functions $(f_1, \ldots, f_k)$, $k \geqslant 2p$ is in *involution* if the GPB vanishes for any subset of $2p$ functions.

Let us also note the following generalization of the Poisson theorem [23].

Theorem 6.1. Let $f_1, \ldots, f_q$, $q \geqslant 2p$ be such that the set of functions $(H_1, \ldots, H_{2p-1}, f_{i_1}, \ldots, f_{i_{2p-1}})$ is in involution (this implies, in particular, that the f_i , $i = 1, \ldots, q$ are constants of motion). Then the quantities $\{f_{i_1}, \ldots, f_{i_{2p}}\}$ are also constants of motion.

Definition 6.5. A set of k functions $c_1(x), \ldots, c_k(x)$ $(1 \leqslant k \leqslant 2p-1)$ will be called a set of k *Casimir functions* if $\{g_1, g_2, \ldots, g_{2p-k}, c_1, \ldots, c_k\} = 0$ for any set of functions $(g_1, g_2, \ldots, g_{2p-k})$.

If one of the Hamiltonians $(H_1, \ldots, H_{2p-1})$ is a Casimir function, then the generalized dynamics defined by (6.1) is trivial. Also, if the set of Hamiltonians contains a Casimir subset, the generalized dynamics will also be trivial (note that if H_1 and H_2 constitute *each* a Casimir subset, the two Hamiltonians (H_1, H_2) will also constitute another, but the reciprocal situation may not be true).

Each Casimir k-subset $(c_1(x), \ldots, c_k(x))$ determines invariant submanifolds of M through the conditions $c_i(x) = c_i$ $(i = 1, \ldots, k)$. The maximal K-subset determines an invariant submanifold of M of minimal dimension, $\dim M - K$, which we may call phase space. Using now the notion of support of an m-skew multivector [24] as the subspace of the space of vector fields generated by the contraction of the multivector with an arbitrary $(m-1)$-form, we make the following conjecture.

Conjecture. The tangent space to the phase space at a point $x \in M$ is the support of $\Lambda(x)$ at that point.

Remark. It is well known that the standard Jacobi identity between f_1, f_2 and H is equivalent to $d\{f_1, f_2\}/dt = \{\dot{f}_1, f_2\} + \{f_1, \dot{f}_2\}$; thus, d/dt is a derivation of the PB. The 'fundamental identity' for Nambu mechanics [1] and its further extensions [6] also corresponds to the existence of a vector field $D_{H_1 \ldots H_{k-1}}$ which is a derivation of the Nambu bracket. In contrast, the vector field (6.3) above is not a derivation of our GPB. It should be noted, however, that having an evolution vector field which is a derivation of a PB is an independent assumption of the associated dynamics and not a necessary one. Nevertheless, the following theorem holds.

Theorem 6.2. Let $H_1, \ldots, H_{2p-1}$ be the 'Hamiltonians' governing the time evolution by (6.1) and let $f_1, \ldots, f_{2p}$ a set of $2p$ functions such that any subset $(f_{i_1}, f_{i_2}, \ldots, f_{i_{2p-1}}, H_{j_1}, \ldots, H_{j_{2p-2}})$ is in involution. Then

$$\frac{d}{dt}\{f_1, \ldots, f_{2p}\} = \{\dot{f}_1, f_2, \ldots, f_{2p}\} + \cdots + \{f_1, \ldots, f_{2p-1}, \dot{f}_{2p}\}\,. \tag{6.4}$$

Proof. It suffices to check that (6.1) in (6.4) leads to an identity on account of the generalized Jacobi identity (5.2). For the case $p = 2$, for instance, the condition of the theorem (see the previous footnote) states that any GPB involving two Hamiltonians and two functions or one Hamiltonian and three functions is zero.

Example. It is well known that Euler's equations describing the free motion of a rigid body around a fixed point are Hamiltonian, $\dot{x}_i = \{H, x_i\}$, where $H \propto a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2$ (where a_i are the principal moments of the body) and the (linear) PS is defined by $\{x_i, x_j\} = \epsilon_{ij}{}^k x_k$ so that $\dot{x}_j \propto \epsilon_{ij}{}^k \partial^i H x_k$, $i, j, k = 1, 2, 3$. The extension of this situation to the motion in a $(2p+1)$-dimensional space provides an example of our GPS. Let the evolution equations be given in terms of $(2p-1)$ Hamiltonians $H_1, \ldots, H_{2p-1}$ (the above case corresponds to $p = 1$) by

$$\dot{x}_j = \epsilon_{i_1 \ldots i_{2p-1} jk} \partial^{i_1} H_1 \cdots \partial^{i_{2p-1}} H_{2p-1} x^k\,, \qquad i, j, k = 1, \ldots, 2p+1\,. \tag{6.5}$$

These equations have the Hamiltonian form (6.1) if the PS is the linear one defined by†

$$\{x_{i_1}, x_{i_2}, \ldots, x_{i_{2p}}\} = \epsilon_{i_1 \ldots i_{2p} k} x^k \equiv \omega_{i_1 \ldots i_{2p}}(x)\,. \tag{6.6}$$

Due to the form of $\omega_{i_1 \ldots i_{2p}}(x)$, it is clear that the GJI (5.4) is trivially fulfilled, since it will always involve the antisymmetrization of repeated indices. Thus, equation (6.6) defines a linear GPS reproducing (6.5); clearly the $(2p-1)$ Hamiltonians are constants of the motion. As in the three-dimensional analogue, the function determining the S^{2p}-sphere $c_{2p} = x_1^2 + \cdots + x_{2p+1}^2$ is a Casimir function (and a constant of motion). Indeed (definition 6.5),

$$\{f_1, \ldots, f_{2p-1}, x_1^2 + \cdots + x_{2p+1}^2\} = 2\omega_{i_1 \ldots i_{2p}} \partial^{i_1} f_1 \cdots \partial^{i_{2p-1}} f_{2p-1} x^{i_{2p}} \tag{6.7}$$

which is zero for all f's on account of (6.6). The trajectory is thus the intersection of the surfaces $H_l = \text{constant}$ $(l = 1, \ldots, 2p-1)$ and $c_{2p} = \text{constant}$.

† It is worth mentioning that the completely antisymmetric tensor of order $(n+1)$ in a $(n+1)$-dimensional vector space gives rise [25] to a Nambu tensor $\epsilon_{i_1 \ldots i_n i_{n+1}} x^{i_{n+1}}$ of order n (i.e. a tensor satisfying the 'fundamental identity' of Nambu mechanics [6]), so that $\omega_{i_1 \ldots i_{2p}}(x)$ above is also a Nambu tensor.

Equations (6.5) are not quadratic in x^i in general and so they do not coincide with the standard Euler equations for the rotation of a higher dimensional rigid body. They become quadratic when $H_1, \ldots, H_{2p-2}$ are linear and H_{2p-1} is a quadratic function of the coordinates, but in this case they reduce to the standard Euler equations in the three-dimensional space determined by the intersection of the $H_i = \text{constant}$ $(i = 1, \ldots, 2p-2)$ hyperplanes.

7. Generalized Poisson structures and differential forms

Let us now rewrite some of the previous expressions in terms of differential forms. First we associate $(n-k)$-forms α with k-skew-symmetric contravariant tensor fields Λ on an n-dimensional orientable manifold M by setting

$$\alpha_\Lambda = i_\Lambda \mu \ , \tag{7.1}$$

where μ stands for a volume form on M (hence, α_Λ depends on the choice of the volume form μ). The mapping $\Psi : \Lambda \mapsto \alpha_\Lambda$ yields an isomorphism between k-skew multivectors and $(n-k)$-forms. For a Λ given by the exterior product of k vector fields $\Lambda = X_1 \wedge \cdots \wedge X_k$ (see equation (4.9)),

$$(i_\Lambda \mu)(Y_1, \ldots, Y_{n-k}) = \mu(X_1, \ldots, X_k, Y_1, \ldots, Y_{n-k}) \, . \tag{7.2}$$

Locally, if for example $\Lambda = \frac{1}{2}\omega^{ij}\partial_i \wedge \partial_j$ and $\mu = dx^1 \wedge \cdots \wedge dx^n$, equation (7.2) gives

$$\alpha_\Lambda = \sum_{i<j} (-1)^{i+j+1} \omega^{ij} dx^1 \wedge \cdots \widehat{dx^i} \cdots \widehat{dx^j} \cdots \wedge dx^n \ , \tag{7.3}$$

where $\widehat{dx^i}$ stands for the omission of dx^i.

For vector fields X, Y,

$$i_{[X,Y]} = i_X d i_Y - i_Y d i_X + i_X i_Y d - d i_{X \wedge Y} \, . \tag{7.4}$$

Similarly, for two bivector fields $\Lambda_1,\ \Lambda_2$,

$$i_{[\Lambda_1, \Lambda_2]} = i_{\Lambda_1} d i_{\Lambda_2} + i_{\Lambda_2} d i_{\Lambda_1} - i_{\Lambda_1} i_{\Lambda_2} d - d i_{\Lambda_1 \wedge \Lambda_2} \, . \tag{7.5}$$

In general, for any two skew-symmetric multivectors A, B of order p, q acting on forms we have†

$$i_{[A,B]} = (-1)^{pq+q} i_A d i_B + (-1)^p i_B d i_A - (-1)^{pq} i_A i_B d - (-1)^{p+q} d i_{A \wedge B} \ , \tag{7.6}$$

from which we find that (7.5) remains valid for any two skew-symmetric multivectors Λ_1 and Λ_2 of *even* order (on a $(p+q-1)$-form α, equation (7.6) reduces to (4.8) since $A \wedge B \in \wedge^{p+q}$). Equation (7.5) now leads to the following theorem which generalizes that in [17] to the arbitrary even-order case:

Theorem 7.1. A Λ defines a GPS if and only if

$$2 i_\Lambda d\alpha_\Lambda = d\alpha_{\Lambda \wedge \Lambda} \, . \tag{7.7}$$

Two GPS $\Lambda_1,\ \Lambda_2$ are compatible if and only if

$$d\alpha_{\Lambda_1 \wedge \Lambda_2} = i_{\Lambda_1} d\alpha_{\Lambda_2} + i_{\Lambda_2} d\alpha_{\Lambda_1} \, . \tag{7.8}$$

† Equation (7.6) is to be compared with the standard formula for vector fields $i_{[X,Y]} = [L_X, i_Y] = i_X d i_Y - i_Y d i_X + i_X i_Y d + d i_X i_Y$ to which it reduces for $p = 1 = q$ (on forms, $i_A i_B = (-1)^{pq} i_B i_A$ and $i_{A\wedge B} = (-1)^{pq} i_{B\wedge A} = (-1)^{pq} i_A i_B$). One could introduce a Lie 'derivative' L_A with respect $A \in \wedge^p(M)$ and rewrite (7.6) in a form similar to the vector field case, namely $i_{[A,B]} = [\![L_A, i_B]\!]$, where $L_A := i_A d + (-1)^{p+1} d i_A$ ($L_A : \wedge^n \to \wedge^{n-p+1}$ and thus it is a derivative only if A is a vector field) and the bracket $[\![\ ,\ ]\!]$ is defined by $[\![L_A, i_B]\!] := (-1)^{q(p+1)} L_A i_B - i_B L_A$.

The isomorphism defined by Ψ suggests that it should be composed in terms of the differential operators $d,\ L_X,\ i_X$ available on forms, so that the properties of the Schouten bracket can be stated in terms of differential forms. As is well known, we have

$$L_X\mu = \mathrm{div}(X)\mu = di_X\mu\ , \tag{7.9}$$

$$d(i_{X\wedge Y}\mu) = i_{[Y,X]}\mu + i_X di_Y\mu - i_Y di_X\mu\ . \tag{7.10}$$

Thus, defining $D = \Psi^{-1}\circ d\circ\Psi$, for the vector fields X, Y we get

$$D(X) = \mathrm{div}(X)\ , \tag{7.11}$$

$$D(X\wedge Y) = -\,\mathrm{div}(X)Y + X\,\mathrm{div}(Y) - [X, Y]\ . \tag{7.12}$$

For contravariant, skew-symmetric tensor fields Λ_1, Λ_2 of arbitrary even order we obtain

$$D(\Lambda_1\wedge\Lambda_2) = D(\Lambda_1)\wedge\Lambda_2 + \Lambda_1\wedge D(\Lambda_2) - [\Lambda_1, \Lambda_2]\ , \tag{7.13}$$

and in the general case we have

$$D(A\wedge B) = (-1)^q D(A)\wedge B + A\wedge D(B) - (-1)^{p+q}[A, B]\ . \tag{7.14}$$

Hence we conclude that if Λ is of arbitrary even order and defines a GPS,

$$D(\Lambda\wedge\Lambda) = 2\Lambda\wedge D(\Lambda)\ . \tag{7.15}$$

We may call a GPS Λ *closed* if $D(\Lambda) = 0$ (which implies $D(\Lambda\wedge\Lambda) = 0$). This is clearly equivalent to the fact that the form α_Λ (and hence $\alpha_{\Lambda\wedge\Lambda}$) is closed. As mentioned, this definition depends on μ so that if μ is replaced by $f\mu$, Λ may no longer be closed.

8. The Schouten–Nijenhuis bracket, GPS and cohomology

Let $\Lambda^{(2p)}$ be a $(2p)$-skew-symmetric multivector defining a GPS as in definition 5.2. Using equation (4.6) it follows that the mapping $\delta_{\Lambda^{(2p)}} : B \mapsto [\Lambda, B]\ ,\ \delta_{\Lambda^{(2p)}} : \wedge^q(M) \to \wedge^{2p+q-1}(M)$ is nilpotent since $[\Lambda, [\Lambda, B]] = 0$. We then have the following theorem.

Theorem 8.1. Let a GPS be defined by $\Lambda^{(2p)}$. The mapping $\delta_{\Lambda^{(2p)}} : B \mapsto [\Lambda, B]$ is nilpotent, $\delta^2_{\Lambda^{(2p)}} = 0$. The operator $\delta_{\Lambda^{(2p)}}$ satisfies (see equations (4.10), (4.6))

$$\delta_{\Lambda^{(2p)}}(B\wedge C) = (\delta_{\Lambda^{(2p)}}B)\wedge C + (-1)^q B\wedge(\delta_{\Lambda^{(2p)}}C) \tag{8.1}$$

$$\delta_{\Lambda^{(2p)}}[B, C] = -[\delta_{\Lambda^{(2p)}}B, C] - (-1)^q[B, \delta_{\Lambda^{(2p)}}C]\ . \tag{8.2}$$

As a result, $\delta_{\Lambda^{(2p)}}$ defines an odd degree cohomology operator; the resulting cohomology will be called *generalized Poisson cohomology*. In particular, for $p = 1$, $\delta_{\Lambda^{(2)}} : \wedge^q(M) \to \wedge^{q+1}(M)$ defines the standard Poisson cohomology [8]; see also [21].

Let us now turn to linear GPS. Let $\mathcal{G}$ be the Lie algebra of a simple compact group G. In this case the de Rham cohomology ring on the group manifold G is the same as the Lie algebra cohomology ring $H_0^*(\mathcal{G}, \mathbb{R})$ for the trivial action. In its Chevalley–Eilenberg version [26] the Lie algebra cocycles are represented by bi-invariant (i.e. left- and right-invariant and hence closed) forms on G (see also, e.g., [27]). The linear standard PS defined by (3.4) is associated (see [11]) with a non-trivial three-cocycle on $\mathcal{G}$ and $[\Lambda^{(2)}, \Lambda^{(2)}] = 0$ (equation (3.5)) is precisely the cocycle condition. This indicates that the linear generalized Poisson structures on $\mathcal{G}^*$ may be found by looking for higher-order Lie algebra cocycles. Let us now show that each of them provides a GPS.

The cohomology ring of any simple Lie algebra of rank l is a free ring generated by l (primitive) forms on G of odd degree $(2m-1)$. These forms are associated with the l

primitive symmetric invariant tensors $k_{i_1\ldots i_m}$ of order m which may be defined on $\mathcal{G}$ and of which the Killing tensor $k_{i_1 i_2}$ is just the first example (and thus $H_0^3(\mathcal{G},\mathbb{R}) \neq 0$ for any simple Lie algebra). As a result, it is possible to associate a $(2m-2)$ skew-symmetric contravariant primitive tensor field linear in x_j to each symmetric invariant polynomial $k_{i_1\ldots i_m}$ of order m. The case $m = 2$ leads to the $\Lambda^{(2)}$ of (3.4), (3.9). For the A_l series ($su(l+1)$), for instance, these forms have order $3, 5, \ldots, (2l+1)$; other orders (but always including 3) appear for the different simple algebras (see, e.g., [27]). Let $\{e_i\}$ be a basis of $\mathcal{G}$. The bi-invariance condition

$$\sum_{s=1}^{q} \omega(e_{i_1}, \ldots, [e_l, e_{i_s}], \ldots, e_{i_q}) = 0 \tag{8.3}$$

reads, in terms of the coordinates $\omega_{i_1\ldots i_q} = \omega(e_{i_1}, \ldots, e_{i_q})$ of the skew-symmetric tensor ω on $\mathcal{G}$ (or, equivalently, LI q-form ω on G),

$$\begin{gathered} C_{li_1}^{\alpha}\omega_{\alpha i_2\ldots i_q} + C_{li_2}^{\alpha}\omega_{i_1\alpha i_3\ldots i_q} + \cdots + C_{li_q}^{\alpha}\omega_{i_1\ldots i_{q-1}\alpha} = 0 \\ \sum_{s=1}^{q} C_{\nu i_s}^{\rho}\omega_{i_1\ldots\widehat{i_s}\rho\ldots i_q} = 0\,. \end{gathered} \tag{8.4}$$

The bi-invariance condition may also be expressed as

$$\epsilon_{i_1\ldots i_q}^{j_1\ldots j_q} C_{\nu j_1}^{\rho}\omega_{\rho j_2\ldots j_q} = 0\,, \tag{8.5}$$

on account of the skew-symmetry of ω. Using the Killing metric this leads to

$$\epsilon_{i_1\ldots i_q}^{j_1\ldots j_q} C_{j_1\rho}^{\nu}\omega^{\rho}{}_{j_2\ldots j_q} = 0\,. \tag{8.6}$$

Let ω be a Lie algebra q-cochain (i.e. a skew-symmetric q-tensor on $\mathcal{G}$ or LI q-form on G). The coboundary operator for the Lie algebra cohomology is given by the following definition.

Definition 8.1a (coboundary operator).

$$(s\omega)(e_{i_1}, \ldots, e_{i_{q+1}}) := \sum_{\substack{s,t=1\\ s<t}}^{q+1} (-1)^{s+t}\omega([e_{i_s}, e_{i_t}], e_{i_1}, \ldots, \widehat{e_{i_s}}, \ldots, \widehat{e_{i_t}}, \ldots, e_{i_{q+1}})\,, \qquad e_i \in \mathcal{G}. \tag{8.7}$$

Thus, in coordinates,

$$\begin{aligned} (s\omega)_{i_1\ldots i_{q+1}} &= \sum_{\substack{s,t=1\\ s<t}}^{q+1} (-1)^{s+t} C_{i_s i_t}^{\rho}\omega_{\rho i_1\ldots\widehat{i_s}\ldots\widehat{i_t}\ldots i_{q+1}} \\ &= \frac{1}{2}\sum_{\substack{s,t=1\\ s<t}}^{q+1} (-1)^{s+t}\epsilon_{i_s i_t}^{j_1 j_2} C_{j_1 j_2}^{\rho}\omega_{\rho i_1\ldots\widehat{i_s}\ldots\widehat{i_t}\ldots i_{q+1}} \\ &= \frac{1}{2}\sum_{\substack{s,t=1\\ s<t}}^{q+1} (-1)^{s+t}\epsilon_{i_s i_t}^{j_1 j_2} C_{j_1 j_2}^{\rho}\frac{1}{(q-1)!}\epsilon_{\rho i_1\ldots\widehat{i_s}\ldots\widehat{i_t}\ldots i_{q+1}}^{j_3\ldots j_{q+1}}\omega_{\rho j_3\ldots j_{q+1}} \\ &= -\frac{1}{2}\frac{1}{(q-1)!} C_{j_1 j_2}^{\rho}\omega_{\rho j_3\ldots j_{q+1}} \sum_{\substack{s,t=1\\ s<t}}^{q+1} (-1)^{s+t+1}\epsilon_{i_s i_t}^{j_1 j_2}\epsilon_{\rho i_1\ldots\widehat{i_s}\ldots\widehat{i_t}\ldots i_{q+1}}^{j_3\ldots j_{q+1}}\,. \end{aligned} \tag{8.8}$$

This provides the equivalent definition:

Definition 8.1b. The action of the coboundary operator on a q-cochain ω is given by

$$(s\omega)_{i_1\ldots i_{q+1}} = -\frac{1}{2}\frac{1}{(q-1)!}\epsilon^{j_1\ldots j_{q+1}}_{i_1\ldots i_{q+1}} C^{\rho}_{j_1 j_2}\omega_{\rho j_3\ldots j_{q+1}}\ ; \qquad s\omega = 0 \quad \text{for} \quad q > r = \dim\mathcal{G}\,. \tag{8.9}$$

As is well known, the invariance condition (8.3) determines a Lie algebra cocycle since, for each fixed j_1, the antisymmetric sum over $j_2, \ldots, j_n$ is zero on account of (8.3). The Poisson structure (3.8) is associated to the structure constants and hence to a three-cocycle. In order to obtain more general structures, we need the expression of the $(2m-1)$-cocycle associated with an order m symmetric tensor on $\mathcal{G}$. This is done in two steps, the first of which is provided by the following lemma.

Lemma 8.1. Let $k_{i_1\ldots i_m}$ be an invariant symmetric polynomial on $\mathcal{G}$ and

$$\tilde{\omega}_{\rho j_2\ldots j_{2m-2}\sigma} := k_{i_1\ldots i_{m-1}\sigma} C^{i_1}_{\rho j_2}\cdots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}}\,. \tag{8.10}$$

Then the odd-order $(2m-1)$-tensor

$$\omega_{\rho l_2\ldots l_{2m-2}\sigma} := \epsilon^{j_2\ldots j_{2m-2}}_{l_2\ldots l_{2m-2}}\tilde{\omega}_{\rho j_2\ldots j_{2m-2}\sigma} \tag{8.11}$$

is a fully skew-symmetric tensor†.

Proof. For $m = 2$, the skew-symmetry of $\omega_{\rho j_2\sigma} = k_{i_1\sigma}C^{i_1}_{\rho j_2}$ is obvious. In general,

$$\begin{aligned}
\epsilon^{j_2\ldots j_{2m-2}}_{l_2\ldots l_{2m-2}}\tilde{\omega}_{\rho j_2\ldots j_{2m-2}\sigma} &= \epsilon^{j_2\ldots j_{2m-2}}_{l_2\ldots l_{2m-2}} k_{i_1\ldots i_{m-1}\sigma}C^{i_1}_{\rho j_2}\cdots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}}\\
&= \epsilon^{j_2\ldots j_{2m-2}}_{l_2\ldots l_{2m-2}}\left[\sum_{s=2}^{m-1} k_{\rho i_2\ldots\widehat{i_s}i_1\ldots i_{m-1}\sigma}C^{i_1}_{j_2 i_s} + k_{\rho i_2\ldots i_{m-1}i_1}C^{i_1}_{j_2\sigma}\right]C^{i_2}_{j_3 j_4}\cdots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}}\\
&= \epsilon^{j_2\ldots j_{2m-2}}_{l_2\ldots l_{2m-2}} k_{\rho i_2\ldots i_{m-1}i_1}C^{i_1}_{j_2\sigma}C^{i_2}_{j_3 j_4}\cdots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}}\\
&= -\epsilon^{j_2\ldots j_{2m-2}}_{l_2\ldots l_{2m-2}} k_{i_1 i_2\ldots i_{m-1}\rho}C^{i_1}_{\sigma j_2}\cdots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}} = -\epsilon^{j_2\ldots j_{2m-2}}_{l_2\ldots l_{2m-2}}\tilde{\omega}_{\sigma j_2\ldots j_{2m-2}\rho}\ ,
\end{aligned}$$

where the invariance of the symmetric tensor k has been used in the second equality, the Jacobi identity in the third and the symmetry of k in the fourth. Since $\omega_{\rho j_2\ldots j_{2m-2}\sigma}$ is skew-symmetric in (ρ, σ) it follows that $\omega_{i_1\ldots i_{2m-1}}$ is a fully skew-symmetric tensor. □

We may then state the following theorem.

Theorem 8.2. The skew-symmetric tensor $\omega_{i_1\ldots i_{2m-1}}$ on $\mathcal{G}$ (or LI $(2m-1)$-form on G) of (8.11) is a $(2m-1)$-cocycle for the Lie algebra cohomology.

Proof. Applying equation (8.9) to (8.11) and using (8.10), it follows that

$$\begin{aligned}
(s\omega)_{i_i\ldots i_{2m}} &= -\frac{1}{2(2m-2)!}\epsilon^{j_1\ldots j_{2m}}_{i_1\ldots i_{2m}}C^{\rho}_{j_1 j_2}\epsilon^{s_3\ldots s_{2m-1}}_{j_3\ldots j_{2m-1}}k_{l_1 l_2\ldots l_{m-1}j_{2m}}C^{l_1}_{\rho s_3}\cdots C^{l_{m-1}}_{s_{2m-2}s_{2m-1}}\\
&= -\frac{(2m-3)!}{2(2m-2)!}\epsilon^{j_1 j_2 s_3\ldots s_{2m-1}j_{2m}}_{i_1\ldots i_{2m}}C^{\rho}_{j_1 j_2}C^{l_1}_{\rho s_3}\cdots C^{l_{m-1}}_{s_{2m-2}s_{2m-1}}k_{l_1 l_2\ldots l_{m-1}j_{2m}}\\
&= 0
\end{aligned}$$

by the Jacobi identity (3.5) in the two first structure constants (indices j_1, j_2, s_3). □

† The origin of (8.11) follows from the fact that given a symmetric invariant polynomial $k_{i_1\ldots i_m}$ on $\mathcal{G}$, the associated skew-symmetric multilinear tensor $\omega_{i_1\ldots i_{2m-1}}$ is

$$\omega(e_{i_1}, \ldots, e_{i_{2m-1}}) = \sum_{s\in S_{(2m-1)}} \pi(s)\, k([e_{s(i_1)}, e_{s(i_2)}], [e_{s(i_3)}, e_{s(i_4)}], \ldots, [e_{s(i_{2m-3})}, e_{s(i_{2m-2})}], e_{s(i_{2m-1})})$$

where $\pi(s)$ is the parity sign of the permutation $s \in S_{(2m-1)}$.

Lemma 8.2. Let $\omega_{i_1\ldots i_q}$, q odd, be an skew-symmetric tensor associated with an invariant symmetric polynomial as above. Then

$$\epsilon^{j_1\ldots j_q}_{i_1\ldots i_q} C^{\rho}_{j_1 j_2} \omega_{\rho j_3\ldots j_q \nu} = 0\,. \tag{8.12}$$

Proof. By equations (8.11) and (8.10)

$$\frac{1}{(q-2)!}\epsilon^{j_1\ldots j_q}_{i_1\ldots i_q} C^{\rho}_{j_1 j_2} \epsilon^{l_3\ldots l_q}_{j_3\ldots j_q} \tilde{\omega}_{\rho l_3\ldots l_q \nu} = \epsilon^{j_1 j_2 l_3\ldots l_q}_{i_1\ldots i_q} C^{\rho}_{j_1 j_2} C^{s_1}_{\rho l_3} \cdots C^{s_p}_{l_{q-1} l_q} k_{s_1\ldots s_p \nu}$$

where $p = (q-1)/2$, which is zero on account of the Jacobi identity for j_1, j_2, l_3. □

The possible cocycles on the different simple Lie algebras are determined by the symmetric invariant polynomials that may be defined on them, which in turn are in one-to-one correspondence with the non-trivial de Rham cocycles, which exist on the corresponding compact group manifolds. Using the above constructions, we may now introduce higher-order (> 2) contravariant skew-symmetric tensors which have zero SNB between themselves.

Let us now apply the results of section 8 to compute the SNB of two contravariant skew-symmetric tensors Ω and Ω' obtained from Lie algebra cocycles,

$$\Omega := \frac{1}{p!}\omega_{i_i\ldots i_p}{}^{\alpha} x_\alpha \partial^{i_1} \wedge \cdots \wedge \partial^{i_p}\,, \qquad \Omega' := \frac{1}{q!}\omega'_{j_i\ldots j_q}{}^{\alpha} x_\alpha \partial^{j_1} \wedge \cdots \wedge \partial^{j_q}\,, \tag{8.13}$$

where $x^\alpha \in \mathcal{G}^*$. Using the Killing metric to raise and lower indices, equation (4.3) gives

$$[\Omega, \Omega']_{i_1\ldots i_{p+q-1}} = \left\{ \frac{1}{(p-1)!q!} \epsilon^{j_1\ldots j_{p+q-1}}_{i_1\ldots i_{p+q-1}} \omega_{\nu j_1\ldots j_{p-1}\alpha} \omega'_{j_p\ldots j_{p+q-1}}{}^{\nu} \right.$$
$$\left. + \frac{(-1)^p}{p!(q-1)!} \epsilon^{j_1\ldots j_{p+q-1}}_{i_1\ldots i_{p+q-1}} \omega'_{\nu j_{p+1}\ldots j_{p+q-1}\alpha} \omega_{j_1\ldots j_p}{}^{\nu} \right\} x^\alpha\,. \tag{8.14}$$

We now state the theorem which gives all the linear GPS on simple Lie algebras.

Theorem 8.3. Let $\mathcal{G}$ be a simple compact† algebra, and let ω and ω' be two non-trivial Lie algebra $(p+1)$- and $(q+1)$-cocycles obtained from the associated $p/2+1$ and $q/2+1$ invariant symmetric tensors on $\mathcal{G}$. Then the associated skew-symmetric contravariant vector fields Ω and Ω' have zero SNB.

Proof. Since both $\Omega \in \wedge^p(\mathcal{G})$, $\Omega' \in \wedge^q(\mathcal{G})$ in (8.14) have arbitrary even order, both terms have the same structure. It is thus sufficient to check that one of them is zero. By equations (8.10) and (8.11) the first term gives

$$\epsilon^{j_1\ldots j_{p+q-1}}_{i_1\ldots i_{p+q-1}} \omega_{\nu j_1\ldots j_{p-1}\alpha} \omega'_{j_p\ldots j_{p+q-1}}{}^{\nu}$$
$$= \epsilon^{j_1\ldots j_{p+q-1}}_{i_1\ldots i_{p+q-1}} \epsilon^{l_1\ldots l_{p-1}}_{j_1\ldots j_{p-1}} C^{s_1}_{\nu l_1} \ldots C^{s_{p/2}}_{l_{p-2} l_{p-1}} k_{s_1\ldots s_{p/2}\alpha} \omega'_{j_p\ldots j_{p+q-1}}{}^{\nu}$$
$$= (p-1)! \epsilon^{l_1\ldots l_{p-1} j_p\ldots j_{p+q-1}}_{i_1\ldots i_{p+q-1}} C^{s_1}_{\nu l_1} \ldots C^{s_{p/2}}_{l_{p-2} l_{p-1}} k_{s_1\ldots s_{p/2}\alpha} \omega'_{j_p\ldots j_{p+q-1}}{}^{\nu} = 0\,, \tag{8.15}$$

which, using (8.6) in the last equality, is zero. □

Since all the even-order skew-symmetric multivector fields Ω associated with the odd-order Lie algebra cocycles have zero SNB between themselves we find the following corollary.

† *Note.* The requirement of compactness is introduced to have a definite Killing–Cartan metric which then may be taken as the unit matrix; this is convenient to identify upper and lower indices.

Corollary 8.1. Let $\mathcal{G}$ be a simple compact algebra, and let $k_{i_1\ldots i_m}$ be a primitive invariant symmetric polynomial of order m. Then the tensor $\omega_{\rho l_2\ldots l_{2m-2}\sigma}$

$$\Omega^{(2m-2)} = \frac{1}{(2m-2)!}\omega_{l_1\ldots l_{2m-2}}{}^{\sigma} x_\sigma \partial^{l_1} \wedge \cdots \wedge \partial^{l_{2m-2}} , \tag{8.16}$$

obtained from the cocycle (8.11), defines a linear GPS on $\mathcal{G}$. In particular (cf equation (3.1))

$$\{x_{i_1}, x_{i_2}, \ldots, x_{i_{2m-2}}\} = \omega_{i_1\ldots i_{2m-2}}{}^{\sigma} x_\sigma , \tag{8.17}$$

where $\omega_{i_1\ldots i_{2m-2}}{}^{\sigma}$ are the 'structure constants' defining the Lie algebra $(2m-1)$-cocycle.

Let Ω be as in equation (8.13) and such that it defines a linear GPS. Then we have the following lemma.

Lemma 8.3. The operator $\partial_\Omega : \wedge^q(\mathcal{G}) \to \wedge^{q+p-1}(\mathcal{G})$ defined by

$$(\partial_\Omega B)_{i_1\ldots i_{p+q-1}} = \frac{1}{p!}\frac{1}{(q-1)!}\epsilon^{j_1\ldots j_{p+q-1}}_{i_1\ldots i_{p+q-1}} \omega_{j_1\ldots j_p}{}^{\nu} B_{\nu j_{p+1}\ldots j_{p+q-1}} , \tag{8.18}$$

where $\Omega^{(2p)}$ is an even skew-symmetric contravariant tensor defining a linear GPS, is nilpotent, $\partial_\Omega^2 = 0$.

Proof. From the definition (8.18) of ∂_Ω,

$$\begin{aligned}
(\partial_\Omega^2 B)_{i_1\ldots i_{2p+q-2}} &= \frac{1}{p!(p+q-2)!}\epsilon^{j_1\ldots j_{2p+q-2}}_{i_1\ldots i_{2p+q-2}} \omega_{j_1\ldots j_p}{}^{\nu} \\
&\quad \times \left(\frac{1}{p!(q-1)!}\epsilon^{k_1\ldots k_{p+q-1}}_{\nu j_{p+1}\ldots j_{2p+q-2}} \omega_{k_1\ldots k_p}{}^{\sigma} B_{\sigma k_{p+1}\ldots k_{p+q-1}} \right) \\
&= \frac{1}{(p!)^2(p+q-2)!(q-1)!} \sum_{s=1}^{p+q-1} (-1)^{s+1} \epsilon^{j_1\ldots j_{2p+q-2}}_{i_1\ldots i_{2p+q-2}} \omega_{j_1\ldots j_p}{}^{k_s} \\
&\quad \times \epsilon^{k_1\ldots \widehat{k_s}\ldots k_{p+q-1}}_{j_{p+1}\ldots j_{2p+q-2}} \omega_{k_1\ldots k_p}{}^{\sigma} B_{\sigma k_{p+1}\ldots k_{p+q-1}} \\
&= \frac{1}{(p!)^2(q-1)!}\left[\sum_{s=1}^{p} (-1)^{s+1} \epsilon^{j_1\ldots j_p k_1 \ldots \widehat{k_s}\ldots k_{p+1}\ldots k_{p+q-1}}_{i_1\ldots i_{2p+q-2}} \omega_{j_1\ldots j_p}{}^{k_s} \right. \\
&\quad \left. + \sum_{s=p+1}^{p+q-1} (-)^{s+1} \epsilon^{j_1\ldots j_p k_1\ldots k_p \ldots \widehat{k_s}\ldots k_{p+q-1}}_{i_1\ldots i_{2p+q-2}} \omega_{j_1\ldots j_p}{}^{k_s} \right] \omega_{k_1\ldots k_p}{}^{\sigma} B_{\sigma k_{p+1}\ldots k_{p+q-1}} \\
&= \frac{p}{(p!)^2(q-1)!}\epsilon^{j_1\ldots j_p k_1\ldots k_{p+q-2}}_{i_1\ldots i_{2p+q-2}} \omega_{j_1\ldots j_p}{}^{\rho} \omega_{\rho k_1\ldots k_{p-1}}{}^{\sigma} B_{\sigma k_p\ldots k_{p+q-2}} \\
&\quad + \frac{(q-1)}{(p!)^2(q-1)!}\epsilon^{j_1\ldots j_p k_1\ldots k_{p+q-2}}_{i_1\ldots i_{2p+q-2}} \omega_{j_1\ldots j_p}{}^{\rho} \omega_{k_1\ldots k_p}{}^{\sigma} B_{\sigma\rho k_{p+1}\ldots k_{p+q-2}} = 0 ,
\end{aligned} \tag{8.19}$$

since in the last equality the first term is zero on account of (8.15), and the second one vanishes since p is even and B is skew-symmetric in (ρ, σ). □

In view of the above lemma and equation (8.14), theorem 8.3 has the following corollary.

Corollary 8.2. If Ω and Ω' (of even order p and p') define two linear GPS, their SNB may be written as

$$[\Omega, \Omega'] = \partial_\Omega \Omega' + \partial_{\Omega'} \Omega .$$

In particular, $\partial_\Omega \Omega = 0$ since Ω is a cocycle for ∂_Ω.

Let us remark that the linear GPS given by the Lie algebra cocycles provide explicit examples of a non-decomposable (i.e. not given by the skew-symmetric product of single vectors) GPS. In contrast, and as conjectured in [25], it has recently been shown [24] that all Nambu-type PS are decomposable (see also [28] for more details on this point). As an example of our theory consider the GPS which may be constructed on $su(3)^*$ defined by

$$\Lambda^{(4)} = \frac{1}{4!}\omega_{i_1 i_2 i_3 i_4}{}^{\sigma} x_\sigma \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_4}} , \qquad \omega_{\rho i_2 i_3 i_4 \sigma} := \frac{1}{2}\epsilon^{j_2 j_3 j_4}_{i_2 i_3 i_4} d_{k_1 k_2 \sigma} C^{k_1}_{\rho j_2} C^{k_2}_{j_3 j_4} , \tag{8.20}$$

where the d_{ijk} are the constants appearing in the anticommutator of the Gell–Mann λ_i matrices, $\{\lambda_i, \lambda_j\} = \frac{4}{3}\delta_{ij} 1_3 + 2d_{ijk}\lambda_k$. It may be checked explicitly that $[\Lambda^{(4)}, \Lambda^{(4)}] = 0$ (see [11] for details). Other examples may be given similarly.

9. Conclusions

In this paper we have established the mathematical basis of a new type of generalized Poisson structures. From a physical point of view, a more detailed investigation of the generalized Hamiltonian dynamics presented here and of its possible applications is needed; clearly, one would like to have more examples besides the simple one provided in section 6. From a mathematical point of view, the linear GPS are also interesting since, when applied to the case of the simple Lie algebras as in the example above, they provide the equivalent of the higher-order Lie algebras [29] which can be defined on any simple Lie algebra associated with its non-trivial cohomology groups. This produces a set of examples (in fact, infinitely many of them: l GPS for *each* simple Lie algebra of rank l, of which the first one is the standard Lie–Poisson structure (3.1) given by the structure constants†), which illustrate our linear GPS in a non-trivial way. The corresponding higher-order simple Lie algebras [29] are in turn special cases of the strongly homotopy Lie algebras (see [30] and references therein) which have been found to be relevant in closed string theory (see, e.g., [31, 32]) and in connection with the Batalin–Vilkovisky antibracket. Nevertheless, more work is needed to see whether the proposed GPS, which are very appealing by virtue of their geometrical contents, also have some direct physical applications.

We conclude here by saying that our analysis could be extended to Lie superalgebras and super-Poisson structures in general by using an appropriate graded version of the SNB. To this end, one first needs a theory of skew graded 'super-multivector' algebras, which to our knowledge is lacking [33]. All these are possible directions for further research.

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† This is in sharp contrast to the Nambu case, where only the structure constants of $sl(2, \mathbb{C})$ may serve as Nambu tensors [25], since those of the other simple algebras do not satisfy the 'fundamental identity' (see also remark 1 in [6]). An interesting question is whether the set of linear GPS based on higher-order Lie algebras and those of the type discussed in the example of section 6 (which trivially satisfy the GJI) constitute all the possible linear GPS.

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